5.6 Notes and Examples

Name: Block: Seat:

L'Hôpital's Rule

1. Warm Up 1: Remember Limits? First substitute to determine if the limit is Type I, II, or III.

   (a) $\lim_{x\to 1} \frac{x^3-1}{x^2+x+1} = \frac{1^3-1}{1^2+1+1} = \frac{0}{3}$ Type I

   (b) $\lim_{x\to 2} \frac{2}{x^2-4} = \frac{2}{2^2-4} = \frac{2}{0}$ Type II

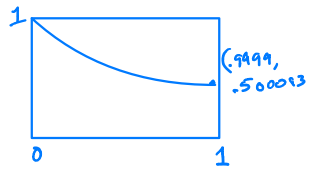
   (c) $\lim_{x\to 2} \frac{x^2+x-6}{x^2-4} = \frac{2^2+2-6}{2^2-4} = \frac{0}{0}$ Type III

2. Warm Up 2: Try these "Type III" limits with a calculator (Recall the Graph and Table Methods?)

   (a) $\lim_{x\to 0} \left(\frac{e^x-1}{x}\right) = 1$

x	$-.01$	$-.001$	$.001$	$.01$
$\frac{e^x-1}{x}$	$.995$	$.9995$	1.0005	1.005

   (b) $\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \frac{1}{2}$



   (c) $\lim_{x\to \infty} \left(1+\frac{2}{x}\right)^x \approx 7.389$

x	999	9999	99999
$1+\frac{2}{x}$	7.3743	7.3876	7.3889

3. Warm Up 2: Try these "Type III" limits by either factoring or multiplying top and bottom with the conjugate:

   (a) $\lim_{x\to -1} \left(\frac{2x^2-2}{x+1}\right)$ $\frac{0}{0}$ $\to \lim_{x\to -1} \frac{2\cancel{(x+1)}(x-1)}{\cancel{(x+1)}} = -4$

   (b) $\lim_{x\to \infty} \left(\frac{3x^2-1}{2x^2+1}\right)$ $\frac{\infty}{\infty}$ $\to \lim_{x\to\infty} \frac{\frac{3\cancel{x^2}}{\cancel{x^2}} - \cancel{\frac{1}{x^2}}^{\,0}}{\frac{2\cancel{x^2}}{\cancel{x^2}} + \cancel{\frac{1}{x^2}}^{\,0}} = \frac{3}{2}$

   (c) $\lim_{x\to 7} \left(\frac{\sqrt{x+2}-3}{x-7}\right)\left(\frac{\sqrt{x+2}+3}{\sqrt{x+2}+3}\right) = \lim_{x\to 7} \frac{(x+2)-9}{(x-7)(\sqrt{x+2}+3)} = \lim_{x\to 7} \frac{\cancel{(x-7)}}{\cancel{(x-7)}\sqrt{x+2}+3} = \frac{1}{3+3}$

   $\frac{0}{0}$

Background of the new method for "Type III" limits: L'Hôpital's Rule

1. Named after Guillaume de L'Hôpital, who published in the first ever differential calculus textbook
2. Actually invented/discovered by Swiss mathematician Johann Bernoulli
3. The method uses derivatives to evaluate indeterminate limits.
4. Don't try it on $\lim_{x\to 3}\left(\frac{2x+7}{4x+1}\right)$. Can you guess why? (This is not an indeterminant form)
   $= \frac{13}{13} = 1$ Type I, Applying L'Hôp, it would be $\frac{1}{2}$!

Now the new method: L'Hôpital's Rule

1. Sometimes, you are not able to simplify equations after doing direct substitution
2. When this happens, it is called an indeterminant form.
3. If $\lim_{x\to c}\frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$
4. In order to use L'Hôpital's Rule you must
   (a) Write the expression in fraction (ratio) form
   (b) State the it is either $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form, and you are using L'Hôpital's rule. (most abbreviations are accepted) L.H, Hospital, etc
   (c) Take the limit of numerator's derivative divided by the denominator's derivative.

4. Examples
   (a) $\lim_{x\to\infty}\frac{x}{e^x}$ $\frac{\infty}{\infty}$ form, by L.H. $= \lim_{x\to\infty}\frac{1}{e^x} = 0$

   (b) $\lim_{x\to\infty}\frac{e^x}{x}$ $\frac{\infty}{\infty}$ form, by L.H. $= \lim_{x\to\infty}\frac{e^x}{1} = +\infty$

   (c) $\lim_{x\to 0}\frac{x}{e^x}$ $= \frac{0}{1} = 0$ (Type I limit)

   L.H. should not be used
   If you did, note what you conclude;
   $\lim_{x\to 0}\frac{1}{e^x} = \frac{1}{1} = 1 \neq 0$

(d) $\lim_{x\to-\infty} x^2e^x$ $\infty \cdot 0$ form ... make it into a ratio
Hint: you need to write this as a fraction

$\lim_{x\to-\infty} \frac{x^2}{e^{-x}}$ $\frac{\infty}{\infty}$ form, by L'Hôp:

$\lim_{x\to-\infty} \frac{2x}{-e^{-x}}$ $\frac{\infty}{\infty}$ form, by L'Hôp:

$\lim_{x\to-\infty} \frac{2}{e^{-x}} = 0$

(e) $\lim_{x\to 2} \frac{x^2+x-6}{x^2-4}$ $\frac{0}{0}$ form, by L'Hôp:

$= \lim_{x\to 2} \frac{2x+1}{2x} = \frac{5}{4}$.

(f) $\lim_{x\to 0} \frac{4e^{2x}-4}{x}$ $\frac{0}{0}$ form, by L'Hôp:

$\lim_{x\to 0} \frac{8e^{2x}}{1} = 8$

5. There are other indeterminate forms that you can use L'Hôpital's Rule with, but you first need to make the expression into a ratio (fractional form)

   1. 1^∞
   2. ∞^0
   3. 0^0
   4. $0 \cdot \infty$
   5. $\infty - \infty$

   (a) 1^∞ form: $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

   Let $y = \left(1+\frac{2}{x}\right)^x$

   $\ln y = x \ln\left(1+\frac{2}{x}\right)$

   $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(1+\frac{2}{x}\right)$

   $= \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{x^{-1}} \rightarrow \frac{0}{0}$ form, by L'Hôp:

   $= \lim_{x\to\infty} \frac{\left(\frac{1}{1+\frac{2}{x}}\right)\left(-2x^{-2}\right)}{-x^{-2}} = \lim_{x\to\infty}\left(\frac{2}{1+\frac{2}{x}}\right) = 2$

   So $\lim_{x\to\infty} y = e^2$

   (b) ∞^0 form: $\lim_{x\to\infty} x^{1/x}$ Let $y = x^{1/x}$ so $\ln y = \frac{1}{x}\ln x = \frac{\ln x}{x}$

   $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x}$ $\frac{\infty}{\infty}$ form, so by L'Hôp:

   $= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

   So $\lim_{x\to\infty} y = e^0 = 1$

   (c) 0^0 form: $\lim_{x\to0^+} x^x$ Let $y = x^x$ so $\ln y = x \ln x$

   $\lim_{x\to0^+} \ln y = \lim_{x\to0^+} x \ln x = \lim_{x\to0^+} \frac{\ln x}{\frac{1}{x}}$ $\frac{-\infty}{\infty}$ form, by L'Hôp:

   $= \lim_{x\to0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} \cdot \frac{x^2}{x^2}$

   $= \lim_{x\to0^+} \frac{x}{-1} = 0$

   So $\lim_{x\to0^+} y = e^0 = 1$

(d) $0 \cdot \infty$ form: $\lim_{x\to\infty} e^{-x}\sqrt{x}$ $= \lim_{x\to\infty} \frac{\sqrt{x}}{e^x}$ $\frac{\infty}{\infty}$ form, by L'Hôp:

$\lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{e^x}$

$\lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^x}$

$\lim_{x\to\infty} \frac{1}{2\sqrt{x}} \cdot \frac{1}{e^x} = 0$

(e) $\infty - \infty$ form: $\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

Get common denom:

$\lim_{x\to 1^+}\left(\frac{(x-1)-\ln x}{\ln x\,(x-1)}\right)$ $\frac{0}{0}$ form, by L'Hôp:

$= \lim_{x\to 1^+} \frac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x}$ $\frac{0}{0}$ form, by L'Hôp:

$= \lim_{x\to 1^+} \frac{-\left(\frac{-1}{x^2}\right)}{\frac{1}{x^2}+\frac{1}{x}} \cdot \frac{x^2}{x^2}$

$= \lim_{x\to 1^+} \frac{1}{1+x} = \frac{1}{2}$

Bonus round (not technically L'Hôpital, but using the same idea about rates):

Using Relative Growth Rates to Evaluate a Limit to $\pm\infty$

When evaluating limits of the the form $\lim_{x\to\infty} \frac{f(x)}{g(x)}$, then the limit is

1. $\pm\infty$ if $f(x)$ grows faster than $g(x)$. (numerator wins)
2. 0 if $f(x)$ grows slower than $g(x)$. (denominator wins)

As $x \to \infty$

$$x^x \succ x! \succ a^x \succ x^a \succ \log_a x$$

6. Examples
   (a) $\lim_{x\to\infty} \frac{e^x}{4^x - 1} = 0$ (Both exponential, but different bases, $e < 4$)
   denom wins!

   (b) $\lim_{x\to\infty} \frac{\ln x}{x^3 + 4} = 0$ ($\ln x$ slower than cubic)

   (c) $\lim_{x\to\infty} \frac{x^3 - 2x + 1}{3x^4 + 3x - 7} = 0$ (Both algebraic, but $x^3 \prec x^4$)

   (d) $\lim_{x\to\infty} \frac{x^x}{x!} = \infty$ ($x^x \succ x!$)